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Pole-free Conditions in Solvable Lattice Models
and their relations to
Determinant Representations of Fusion transfer matrices
- Solution to a certain family of discrete Toda field equations-

Junji Suzuki*
Institut für Theoretische Physik, Universität zu Köln**
Züplicher Strasse 77, D50937 Köln, Germany

1 Introduction

In soliton theories, tau functions deserve the most fundamental objects. They can be , in most cases, represented by determinants. For the KP hierarchy their appearance is quite naturally explained by the Sato theory.

Recently, several discrete equations are proposed which possess such determinantal structure for their tau functions. Some of them have an interesting property which is claimed to be a discrete analogue to the Painlevé property, and thus often referred to as discrete Painlevé equations (DPE)[GRP, RGH,Kaji]. The property, singularity confinement, still remains as mystery and its relation to determinant representations for tau functions is yet to be understood.

On the other hand, we proposed, in previous publications, a certain family of discrete equations (T-system) which also have determinantal expressions for solutions.

They have a background in solvable lattice models in two dimensions: They come from functional identities among commuting transfer matrices. The Hirota-Miwa equation is a specialization to A_r type. We will show, moreover, in the section 2 they are discrete analogues to the Toda field equations based on classical Lie algebras.

We seek a specific solution being pole-free. This demand comes from the origin. The field variables should be identified with eigenvalues of transfer matrices. They must be non-singular as the original Boltzmann weights are so. This pole-free condition strongly restricts the form of solution to T-system so that it takes a determinantal structure. It must be stressed that this type of discrete Toda field equation yields natural reasoning for determinantal expressions. The determinantal forms so obtained are of course solutions

*E-mail address: js@sun30.thp.uni-koeln.de
**On leave of absence from Institute of Physics, University of Tokyo, Komaba

to T-system even if one forgets about pole-free condition (the Bethe ansatz equation, BAE), although they are no longer analytic. In this sense, BAE is a hidden reason for the structure. Certain similarities between some sets of DPE and T-systems make us expect that it would be also possible for the former to unveil a hidden reason.

This paper is organized as follows.

In the next section, we present the T-systems for A_r and B_r. In passing to the continuum limit, we will show that they coincide with the Toda field equations. we give the answer to the T-system by introducing generalized determinantal expressions in section 3. A "quantum analogue" to Jacobi-Trudi and Giambelli identities will be discussed. Until this stage, we do not touch our guiding principle in finding such expressions.

In the rest of sections, we will give some elementary backgrounds of solvable lattice models. The fusion procedure lies in heart of the T-system. We will explain this by adopting simple examples in section 4. The sections 5 is devoted to the review of a "modified" analytic Bethe ansatz, which is the main source in obtaining various combinatorial expressions. We will discuss the equivalence between these combinatorial objects and determinant expressions in section 6.

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2. $T-$ system as a discrete Toda field equation

As remarked in the introduction, T-system is a set of functional relations among transfer matrices. We present only the resultant equations without referring to their origin. Our attitude here is to regard them as a kind of discrete evolution equations. We will present some backgrounds in later sections for reference.

T-system exists for arbitrary Lie algebras. Here we focus on those for A_r and B_r,

A_r :

$$T_m^{(a)}(u-1)T_m^{(a)}(u+1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + g_m^{(a)}(u)T_m^{(a-1)}(u)T_m^{(a+1)}(u) \qquad 1 \le a \le r, \tag{2.1}$$

B_r :

$$T_m^{(a)}(u-1)T_m^{(a)}(u+1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + g_m^{(a)}(u)T_m^{(a-1)}(u)T_m^{(a+1)}(u)$$

$$1 \le a \le r-2, \tag{2.2a}$$

$$T_m^{(r-1)}(u-1)T_m^{(r-1)}(u+1) = T_{m+1}^{(r-1)}(u)T_{m-1}^{(r-1)}(u) + g_m^{(r-1)}(u)T_m^{(r-2)}(u)T_{2m}^{(r)}(u) \tag{2.2b}$$

$$T_{2m}^{(r)}(u-\frac{1}{2})T_{2m}^{(r)}(u+\frac{1}{2}) = T_{2m+1}^{(r)}(u)T_{2m-1}^{(r)}(u) + g_{2m}^{(r)}(u)T_m^{(r-1)}(u-\frac{1}{2})T_m^{(r-1)}(u+\frac{1}{2}), \tag{2.2c}$$

$$T_{2m+1}^{(r)}(u-\frac{1}{2})T_{2m+1}^{(r)}(u+\frac{1}{2}) = T_{2m+2}^{(r)}(u)T_{2m}^{(r)}(u) + g_{2m+1}^{(r)}(u)T_m^{(r-1)}(u)T_{m+1}^{(r-1)}(u) \tag{2.2d}$$

where $T_m^{(0)}(u) = T_0^{(a)}(u) = 1$ and $g_m^{(a)}(u)$'s are arbitrary functions satisfying,

$$g_m^{(a)}(u+1/t_a)g_m^{(a)}(u-1/t_a) = g_{m+1}^{(a)}(u)g_{m-1}^{(a)}(u) \quad a = 1,\cdots,r, \tag{2.3}$$

in which t_a's are given by 1 except for $a = r, B_r$ $(t_r = 2)$.
Here $m \in Z$ runs over either infinite or finite set which we do not specify here. See the discussion in section 3 of [KNS 1]
Several "good" properties of these equations have been observed[KNS1].

(1) In $|u| \to \infty$, these equations reduce to those for corresponding Yangian characters.
(2) Solving the T-system recursively, we empirically find that $T_m^{(a)}(u)$ is a polynomial in terms of $T_1^{(b)}(u + \text{shift}), (b = 1,\cdots,r)$, and moreover it can be represented as a determinant of a certain sparse matrix. *

We here add one important property in a "continuum limit"[KOS].
One can freely re-scale $u \in C$ so that the difference expression above reduces to differential one w.r.t. u:

$$T_m^{(a)}(u+n) \to T_m^{(a)}(\frac{u}{\epsilon}+n) \to (\phi_m^{(a)}(u) + n\epsilon\partial_u\phi_m^{(a)}(u) + \cdots). \tag{2.4}$$

where $\phi_m^{(a)}(u)$ is renormalized $T_m^{(a)}(u/\epsilon)$.
We tentatively assume m's to be rational numbers and take a similar limit. As $g_m^{(a)}(u)$'s are arbitrary function with one constraint eq(2.3), we can re-scale them as $O(\epsilon^2)$. The resultant equations have a unified expression with two continuous "space-time" variables u, m and one discrete index $a = 1,\cdots,r$.

$$(\partial_u^2 - \partial_m^2)\psi_a(u,m) = \text{prefactor} \exp(-\sum_{b=1}^{r} A_{ab}\psi_b(u)) \tag{2.5}$$

where $\psi_a(u,m)$ is a re-scaled logarithm of $\phi_m^{(a)}(u)$ and $A_{ab} = \frac{2(\alpha_a|\alpha_b)}{(\alpha_a|\alpha_a)}$ is the Cartan matrix. The prefactor is a re-scaled $g_m^{(a)}(u)$ which can be either a function of space-time variables

* The conjectured sparse matrix expressions in [KNS1] are now proved in [KNH]

or mere a constant. This is nothing but the Toda field equation for the Lie algebra specified by A_{ab}. T-system corresponding to an arbitrary classical Lie algebra reduces to eq(2.5) in appropriate limits. In this sense, we regard T-system as a *discrete Toda field equation*. In the next section, we will give the solutions to T-system for given "initial data" $T_1^{(a)}(u), a = 1, \cdots, r$. And these will justify our empirical rule (2) above.

3. Solution to T-system

3.1 *Non-Commutative Generating Functions*

Symmetric functions play an important role in soliton theories. We thus expect certain analogues in discrete cases. This is indeed the case for T-system as we will see in the following.

Let us introduce two fundamental quantities $T^a(u)$, $(a = 1, 2, \cdots)$ and $T_m(u)$, $(m = 1, 2, \cdots)$. They are analogues to bases for symmetric functions, and defined by generating series. We prepare a set of functions, $x_a^A(u), (a = 1, \cdots r)$ and $x_a^B(u), x_0^B(u), x_{\bar{a}}^B(u)(a = 1, \cdots r)$ for the A_r type and the B_r type T-system, respectively. Their explicit forms are not necessary for the time being.

We introduce "non-commutative generating series":

$$\sum_{a=0}^{\infty} T^a(u+a-1)X^a = (1 + x_r^A(u)X) \cdots (1 + x_1^A(u)X) \tag{3.1a}$$

$$\sum_{m=0}^{\infty} T_m(u+m-1)X^m = (1 - x_1^A(u)X)^{-1} \cdots (1 - x_r^A(u)X)^{-1}, \tag{3.1b}$$

for A_r,

$$\begin{aligned}\sum_{a=0}^{\infty} T^a(u+a-1)X^a =& (1 + x_1^B(u)X) \cdots (1 + x_{\bar{r}}^B(u)X)(1 - x_0^B(u)X)^{-1} \\ & (1 + x_r^B(u)X) \cdots (1 + x_{1}^B(u)X)\end{aligned} \tag{3.2a}$$

$$\begin{aligned}\sum_{m=0}^{\infty} T_m(u+m-1)X^m =& (1 - x_1^B(u)X)^{-1} \cdots (1 - x_r^B(u)X)^{-1}(1 + x_0^B(u)X) \\ & (1 - x_{\bar{r}}^B(u)X)^{-1} \cdots (1 - x_{\bar{1}}^B(u)X)^{-1}\end{aligned} \tag{3.2b}$$

for B_r.

We use same symbol $T^a(u)$ for both algebras. There should be no confusion.

In the above X is an operator acting on an arbitrary function $A(u)$ by,

$$XA(u) = A(u+2)X. \tag{3.3}$$

X can be given by differential operator, $X = \exp(2\partial_u)$. Namely, it exponentiates the momentum operator conjugate to the coordinate u. Taking a "classical limit", i.e., commutative limit, the eqs(3.1a,b). reduce to the classical generating relations written in many references, while eqs(3.2a,b) seems novel. See e.g.,[Mac].
We then make the following identification between some quantities in $T-$ system and those in the above series,

$$T_1^{(a)}(u) = T^a(u), \qquad a = 1, \cdots, r(\text{ for } A_r), r-1(\text{ for } B_r) \tag{3.4a}$$

$$T_m^{(1)}(u) = T_m(u), \qquad m = 1, \cdots. \tag{3.4b}$$

For B_r case, one further subsidiary condition should be imposed,

$$T^a(u) + T^{2r-1-a}(u) = T_1^{(r)}(u - r + a + \frac{1}{2})T_1^{(r)}(u + r - a - \frac{1}{2}) \quad \forall a \in \mathbf{Z}. \tag{3.5}$$

Note that the relation is invariant under the exchange $a \leftrightarrow 2r-1-a$. If $a < 0$ or $a > 2r-1$, there is in fact only one term on the LHS. Though the relation seems somewhat strange, it has a sound ground in solvable lattice models.[Okado, KS]

3.2 *Quantum Jacobi-Trudi and Giambelli Formulae*

Rather than dealing with T-system solution alone, we find it convenient to introduce slightly generalized objects[KOS]. Let us prepare notations. Let $\mu = (\mu_1, \mu_2, \ldots)$, $\mu_1 \geq \mu_2 \geq \cdots \geq 0$ be a Young diagram and $\mu' = (\mu'_1, \mu'_2, \ldots)$ be its transpose. We denote by d_μ, the length of the main diagonal of μ. A skew-Young diagram means a pair of Young diagrams $\lambda \subset \mu$ satisfying $\mu_i \geq \lambda_i$ for $\forall i$. We associate a function $T_{\lambda\subset\mu}(u)$ to a skew Young tableaux $\lambda \subset \mu$.
At this stage, we have to notice different properties of A_r and B_r. For the latter algebra, the most fundamental is the spin representation. Consequently, we have to distinguish between a representation containing odd-spin representation and one with even-spin representation. See [KOS] for the detail.
For A_r or the spin-even case of B_r, the explicit form of $T_{\lambda\subset\mu}(u)$ is given by

$$T_{\lambda\subset\mu}(u) = det \begin{pmatrix} 0 & \cdots & 0 & R_{11} & \cdots & R_{1d_\mu} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & R_{d_\lambda 1} & \cdots & R_{d_\lambda d_\mu} \\ C_{11} & \cdots & C_{1d_\lambda} & H_{11} & \cdots & H_{1d_\mu} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ C_{d_\mu 1} & \cdots & C_{d_\mu d_\lambda} & H_{d_\mu 1} & \cdots & H_{d_\mu d_\mu} \end{pmatrix}, \tag{3.6a}$$

where

$$
\begin{aligned}
R_{ij} &= T_{\mu_j-\lambda_i+i-j}(u+\mu'_1-\mu_1+\mu_j+\lambda_i-i-j+1),\\
C_{ij} &= -T^{\mu'_i-\lambda'_j-i+j}(u+\mu'_1-\mu_1-\mu'_i-\lambda'_j+i+j-1),\\
H_{ij} &= T_{\mu'_i-i,\mu_j-j}(u+\mu'_1-\mu_1-\mu'_i+\mu_j+i-j).
\end{aligned}
\tag{3.6b}
$$

$T_{k,\ell}(u)$ is the function corresponding to a hook, $T_{(\ell+1,1^k)}$.
Two particular cases corresponding to the formal choices $\mu_i = \lambda_i$ or $\mu'_i = \lambda'_i$ for $1 \le i \le d_\lambda = d_\mu$ yield simpler formulae. In these cases, redefining μ_i, μ'_i, λ_i and λ'_i so that $\lambda'_{\mu_1} = \lambda_{\mu'_1} = 0$, we have

$$
\begin{aligned}
T_{\lambda\subset\mu}(u) &= det_{1\le i,j\le \mu_1}(T^{\mu'_i-\lambda'_j-i+j}(u+\mu'_1-\mu_1-\mu'_i-\lambda'_j+i+j-1)), &(3.7a)\\
&= det_{1\le i,j\le \mu'_1}(T_{\mu_j-\lambda_i+i-j}(u+\mu'_1-\mu_1+\mu_j+\lambda_i-i-j+1)). &(3.7b)
\end{aligned}
$$

The hook function in eq(3.6) is thus expressible in T^a and T_m.
The above relations are quantum analogues to Jacobi-Trudi or Giambelli formulae for Schur functions, in the sense that they reduce to "classical" ones dropping $u-$ dependency. They admit a unified description for A_r and B_r with spin-even cases, although their fundamental elements , $T^a(u), T_m(u)$ are defined by different generating series.
Next we consider the spin-odd case of B_r. An element in the spin representation can be also labeled by a column of r-boxes with appropriate letters in them. We only treat the case where Young diagram μ has spin representation in the left most column. Let us denote the corresponding function by $S^L_{\lambda\subset\mu}(u)$,

$$
\begin{aligned}
S^L_{\lambda\subset\mu}(u) &= det_{1\le i,j\le \mu_1}(\mathcal{S}^L_{ij}) &(3.8a)\\
&= det_{1\le i,j\le \mu'_2}(\overline{\mathcal{S}}^L_{ij}), &(3.8b)
\end{aligned}
$$

where

$$
\mathcal{S}^L_{ij} = \begin{cases} T^{\mu'_j-\lambda'_i+i-j}(u+2\mu'_1-\mu'_j-\lambda'_i+i+j-r-\frac{5}{2}) & j\ge 2\\ T^{(r)}_1(u+2i-2+2(\mu'_1-\lambda'_i-r)) & j=1 \end{cases}, \tag{3.9a}
$$

$$
\overline{\mathcal{S}}^L_{ij} = \begin{cases} T_{\mu_i-\lambda_j-i+j}(u+2\mu'_1+\mu_i+\lambda_j-i-j-r-\frac{1}{2}) & 1\le j\le \lambda'_1\\ \mathcal{H}^L_{\mu_i+\lambda'_1-i}(u+2\mu'_1-2\lambda'_1-2r) & j=\lambda'_1+1\\ T_{\mu_i-i+j-1}(u+2\mu'_1+\mu_i-i-j-r+\frac{1}{2}) & j>\lambda'_1+1 \end{cases}, \tag{3.9b}
$$

$$
\mathcal{H}^L_m(u) = \sum_{l=0}^{m}(-1)^l T^{(r)}_1(u+2l)T_{m-l}(u+m+r+l-\frac{1}{2}). \tag{3.9c}
$$

We also have its "dual" where a Young diagram has a spin representation in its rightmost column, which might as well be skipped here.

3.3 *Outline of Proof*

The functions $T_{\lambda\subset\mu}(u)$ (3.6) and $S^L_{\lambda\subset\mu}(u)$ (3.8) provide solutions to the T-system for A_r or B_r. For $m \in \mathbf{Z}_{\geq 0}$, put

$$T^{(a)}_m(u) = T_{(m^a)}(u) \tag{3.10}$$

for $1 \leq a \leq r(A_r)$ and for $1 \leq a \leq r-1(B_r)$. We have two further identifications for B_r

$$T^{(r)}_{2m}(u) = T_{(m^r)}(u), \tag{3.11a}$$

$$T^{(r)}_{2m+1}(u) = S^L_{((m+1)^r)}(u-m). \tag{3.11b}$$

Then eqs(2.1),(2.2a,b) reduce to the Jacobi equality, as discussed in [KNS1] for only A_r models. Eqs(2.2 c, d) need novel insights because of spin-odd terms. We sketch the proof for eq(2.2c), and leave the proof of eq(2.2d) to readers.

Proof of eq(2.2c)

Let us slightly deform eq(2.2c) into,

$$T^{(r)}_{2m+1}(u)T^{(r)}_{2m-1}(u) = T^{(r)}_{2m}(u-\frac{1}{2})T^{(r)}_{2m}(u+\frac{1}{2}) - T^{(r-1)}_{m}(u-\frac{1}{2})T^{(r-1)}_{m}(u+\frac{1}{2}). \tag{2.2c'}$$

By specializing formula (3.8) to the case, $\lambda = \phi, \mu = ((\ell+1)^r)$, we find the $(\ell+1)\times(\ell+1)$ matrix determinant expression,

$$\begin{aligned} T^{(r)}_{2\ell+1}(u) &= \det\begin{vmatrix} T^{(r)}_1(u-\ell) & T^{r-1}(u-\ell+\frac{1}{2}) & \dots & T^{r-\ell}(u-\frac{1}{2}) \\ T^{(r)}_1(u-\ell+2) & T^{r}(u-\ell+\frac{3}{2}) & \dots & T^{r-\ell+1}(u+\frac{1}{2}) \\ \vdots & \vdots & & \vdots \\ T^{(r)}_1(u+\ell) & T^{r+\ell-1}(u+\frac{1}{2}) & \dots & T^{r}(u+\ell-\frac{1}{2}) \end{vmatrix}. \\ &= \sum_{j=0}^{\ell}(-1)^j T^{(r)}_1(u-\ell+2j)R^{(\ell)}_j(u), \end{aligned} \tag{3.12}$$

where $R^{(\ell)}_j(u)$ signifies the determinant of the $(j,1)$-minor. With use of this form for $T^{(r)}_{2m\pm1}$ and applying eq. (3.5), we can rewrite the lhs of (2.2c') as

$$\begin{aligned} lhs = &\sum_{i=0}^{m-1}\sum_{j=0}^{m}(-1)^{i+j}R^{(m-1)}_i(u)R^{(m)}_j(u) \\ &\left\{T^{r+j-i-1}(u-m+i+j+\frac{1}{2}) + T^{r+i-j}(u-m+i+j+\frac{1}{2})\right\}. \end{aligned} \tag{3.13}$$

One sums up the first (second) term in the rhs of(3.13) w.r.t. $j(i)$. The result can be rewritten as the sums of determinants.

$$\sum_{i=0}^{m-1}(-1)^i R^{(m-1)}_i(u)\det\begin{vmatrix} T^{r-i-1}_1(u-m+i+\frac{1}{2}) & T^{r-1}(u-m+\frac{1}{2}) & \dots & T^{r-m}(u-\frac{1}{2}) \\ T^{r-i}_1(u-m+i+\frac{3}{2}) & T^{r}(u-m+\frac{3}{2}) & \dots & T^{r-m+1}(u+\frac{1}{2}) \\ \vdots & \vdots & & \vdots \\ T^{r-i+m-1}_1(u+i+\frac{1}{2}) & T^{r+m-1}(u+\frac{1}{2}) & \dots & T^{r}(u+m-\frac{1}{2}) \end{vmatrix}$$

$$+\sum_{j=0}^{m}(-1)^j R_i^{(m-1)}(u)\det\begin{vmatrix} T_1^{r-j}(u-m+j+\frac{1}{2}) & T^{r-1}(u-m+\frac{3}{2}) & \dots & T^{r-m}(u-\frac{1}{2}) \\ T_1^{r-j+1}(u-m+j+\frac{3}{2}) & T^{r}(u-m+\frac{5}{2}) & \dots & T^{r-m+1}(u+\frac{1}{2}) \\ \vdots & \vdots & & \vdots \\ T_1^{r-j+m-1}(u+j-\frac{1}{2}) & T^{r+m-2}(u+\frac{1}{2}) & \dots & T^{r}(u+m-\frac{3}{2}) \end{vmatrix}. \tag{3.14}$$

Apparently, the first summation vanishes whereas $j=0$ and $j=m$ terms in the second summation contribute. By noticing $R_0^{(m)} = T_{(m^r)}(u+\frac{1}{2}) = T_{2m}^{(r)}(u+\frac{1}{2})$, $R_m^{(m)} = T_m^{r-1}(u-\frac{1}{2})$, and using the above explicit forms, we can easily establish that eq(3.14) coincides with the rhs of (2.2c'). □

We remark that the above proof does not utilize any properties of underlying solvable models except for (3.5). The determinantal structure of solution seems to be accidental just like in the other discrete equations, if one does not have any knowledge on the background. The rest of this paper will be devoted to present reasoning from solvable lattice models in two dimensions.

4 Fusion hierarchy of vertex models

We consider vertex models on a square lattice. Physical degrees of freedom are assigned to horizontal and vertical edges. To each vertex, we associate a Boltzmann weight , or an element in $R-$ matrix according to four physical variables on edges surrounding the vertex. To define a vertex model, therefore, we need to specify what kind of space physical variables belong to. Fusion procedure is the most fundamental technique in obtaining models defined on "composite" spaces out of "fundamental" ones. Roughly speaking, the two indices a, m appeared in the field variable T indicate how many times one employs fusion procedure for defining the model. And $T-$ system itself might be a consequence of this. Thus it might be meaningful to present an elemantary explanation to the fusion procedure here.

Let us take a A_r vertex model as an example. For this model, the space of vector representation, V_{Λ_1} deserves most fundamental. Its elements ,i.e.,physical variables in that space can be labeled by integers $1,\cdots,r$. The $R-$ matrix acting on $V_{\Lambda_1}\otimes V_{\Lambda_1}$ is given by

$$R_{V_{\Lambda_1},V_{\Lambda_1}}(u) = \sum_{\alpha=1,\cdots,r}(1+u/2)E_{\alpha\alpha}\otimes E_{\alpha\alpha} + \sum_{\alpha\neq\beta}E_{\alpha\beta}\otimes E_{\beta\alpha} + \sum_{\alpha\neq\beta}u/2E_{\alpha\alpha}\otimes E_{\beta\beta} \tag{4.1}$$

where $E_{\alpha\beta}$ is a matrix element, $(E_{\alpha\beta})_{i,j} = \delta_{\alpha,i}\delta_{\beta,j}$. We regard R-matrix as an operator sending the "bottom" ("left") state to the "upper" ("right") state, graphically.

This R-matrix satisfies the Yang-Baxter relation:

$$R_{V_2V_3}(u)R_{V_1V_3}(u+v)R_{V_1V_2}(v) = R_{V_1V_2}(v)R_{V_1V_3}(u+v)R_{V_2V_3}(u). \tag{4.2}$$

Let P be a permutation operator in $V_{\Lambda_1} \otimes V_{\Lambda_1}$, $P(a \otimes b) = b \otimes a$. We define $R^{\vee}(u) = PR(u)$. It yields decomposition of the tensorial space as,

$$R^{\vee}(u) = (1 + u/2)P_{2\Lambda_1} + (1 - u/2)P_{\Lambda_2}. \tag{4.3}$$

where P_{Λ} denotes the projector to the highest weight module V_{Λ}. Here we see two singular points in $R^{\vee}(u)$, $u = -2$ and 2 where it reduces to a projector to a subspace in $V_{\Lambda_1} \otimes V_{\Lambda_1}$. We introduce operators $R_{<V_1,V_2>,V_3}(u)$, $R_{(V_1,V_2),V_3}(u)$ acting on $V_1 \otimes V_2 \otimes V_3$ where $V_i = V_{\Lambda_1}$,

$$R_{<V_1,V_2>,V_3}(u) = \frac{1}{2} R^{\vee}_{V_1,V_2}(-2) R_{V_1,V_3}(u+1) R_{V_1,V_2}(u-1) \tag{4.4a}$$

$$R_{(V_1,V_2),V_3}(u) = \frac{1}{2} R^{\vee}_{V_1,V_2}(2) R_{V_1,V_3}(u-1) R_{V_1,V_2}(u+1). \tag{4.4b}$$

Pictorially, we regard the third space as the vertical one, and the first and second as horizontal ones. Thanks to the Yang Baxter relation and $(\frac{1}{2}R^{\vee}(\pm 2))^2 = \frac{1}{2}R^{\vee}(\pm 2)$, we have

$$R_{<V_1,V_2>,V_3}(u) = \frac{1}{2} R^{\vee}_{V_1,V_2}(-2) R_{V_1,V_3}(u-1) R_{V_1,V_2}(u+1) \frac{1}{2} R^{\vee}_{V_1,V_2}(-2) \tag{4.5a}$$

$$R_{(V_1,V_2),V_3}(u) = \frac{1}{2} R^{\vee}_{V_1,V_2}(2) R_{V_1,V_3}(u+1) R_{V_1,V_2}(u-1) \frac{1}{2} R^{\vee}_{V_1,V_2}(2). \tag{4.5b}$$

Now that they have projectors in both horizontal ends, we can regard $R_{<V_1,V_2>,V_3}(u)$, $R_{(V_1,V_2),V_3}(u)$ as operators acting on $V_{\Lambda_2} \otimes V_{\Lambda_1}$ and $V_{2\Lambda_1} \otimes V_{\Lambda_1}$, respectively. Similarly one can build $R-$matrix acting on far more composite spaces labeled by skew Young diagrams, in principle. We will not go into details. We content ourselves by demonstrating the first equation of the $T-$system can be easily derived from the above argument. Let the monodromy matrix $J_{V_0}(u)$ acting on space $\mathcal{V} = (V_1 \otimes V_2 \otimes \cdots \otimes V_{N-1} \otimes V_N)$ be

$$J_{V_0}(u) = R_{V_0,V_1}(u) R_{V_0,V_2}(u) \cdots R_{V_0,V_{N-1}}(u) R_{V_0,V_N}(u). \tag{4.6}$$

$\mathcal{V}$ is referred to as a quantum space.
The transfer matrix $T_{V_0}(u)$is given by the trace:

$$T_{V_0}(u) = Tr_{V_0} J_{V_0}(u). \tag{4.7}$$

Note that we label the transfer matrix by its horizontal space. We adopt this convention hereafter.

Let $V_0 = V_0' = V_{\Lambda_1}$ and prepare two transfer matrices stacked vertically. We use the fact that the identity operator in $V_{\Lambda_1} \otimes V_{\Lambda_1}$ is $\frac{1}{2}(R^\vee(2) + R^\vee(-2))$. Inserting this, we have

$$\begin{aligned}
T_{\Lambda_1}(u+1)T_{\Lambda_1}(u-1) &= Tr_{V_0\otimes V_0'}\frac{1}{2}(R^\vee_{V_0,V_0'}(2) + R^\vee_{V_0,V_0'}(-2))J_{V_0}(u+1)J_{V_0'}(u-1) \\
&= Tr_{V_0\otimes V_0'}\Big(\frac{1}{2}R^\vee_{V_0,V_0'}(-2)J_{V_0}(u+1)J_{V_0'}(u-1) + J_{V_0}(u+1)J_{V_0'}(u-1)\frac{1}{2}R^\vee_{V_0,V_0'}(2)\Big) \\
&= Tr_{V_0\otimes V_0'}\Big(R_{<V_0,V_0'>V_1}(u)R_{<V_0,V_0'>V_2}(u)\cdots R_{<V_0,V_0'>V_N}(u) \\
&+ R_{(V_0,V_0')V_1}(u)R_{(V_0,V_0')V_2}(u)\cdots R_{(V_0,V_0')V_N}(u)\Big) \\
&= T_{V_{\Lambda_2}}(u) + T_{V_{2\Lambda_1}}(u)
\end{aligned} \tag{4.8}$$

where we use the cyclic property of trace and the Yang Baxter relation repeatedly. Note that the above modules can be labeled by boxes such that $T_{\Lambda_1} = T_{(1)}, T_{\Lambda_2} = T_{(1^2)}$ and $T_{2\Lambda_1} = T_{(2)}$. Then eq.(4.8) is nothing but the first of T-system of A_r type under the identification (3.10) and our convention, $T^{(0)}_m(u) = T^{(a)}_0 = 1$. T-system may be derived in this way in principle, although it was proposed via different route.[KNS1]
Before closing this section, we make an important remark. We call a set of models a fusion hierarchy if they are off-springs of a fundamental model. As a consequence of fusion procedure, transfer matrices belonging to a hierarchy and sharing a same quantum space commute with each other:

$$[T_\mu(u), T_{\mu'}(u')] = 0. \tag{4.9}$$

Consequently, they can be treated as scalars on the space of their common eigenstates. We do not , in what follows, distinguish a transfer matrix from its eigenvalue in this sense.

5 "Bethe-strap procedure" and Yangian analogue of Young tableaux

To solve an eigenvalue equation analytically, one usually builds eigenfunctions at the same time. The latters are generally involved and irrelevant if only eigenvalues are of interest. Lanczos method would be a good example. In the content of solvable lattice models, this is first noticed by Baxter[Bax] and emphasized by Reshetikhin[Res] as the analytic Bethe ansatz method. Here we further proceed and propose a modified version of their ansatz so that it can be applied to wider range of models[KS]. Let us take the six vertex model as the simplest example. The $R-$ matrix is $r = 2$ specialization in eq(4.1). The transfer matrix for this case was diagonalized long ago by Lieb[EL] constructing explicit eigenfunctions. His result reads,

$$T_1(u) = \Big(\frac{(2+u)}{2}\Big)^N\frac{Q(u-1)}{Q(u+1)} + \Big(\frac{u}{2}\Big)^N\frac{Q(u+3)}{Q(u+1)}. \tag{5.1}$$

where

$$Q(u) = \prod_{j=1}^{m} (u - iu_j)$$

and u_j's are parameters which label different eigenvalues.
From the original R–matrix , the resultant eigenvalue should be obviously pole-free. However the first and second terms in eq.(5.2) seem to have poles at $u = -1+iu_k(k = 1, \cdots, m)$. These singularities must cancel with each other. Therefore we demand the residues at these point should be zero, i.e.,

$$\begin{aligned}\Big(\frac{(iu_k+1)}{(iu_k-1)}\Big)^N &= -\frac{Q(iu_k+2)}{Q(iu_k-2)} \\ &= -\prod_{j=1}^{m} \frac{(iu_k - iu_j + 2)}{(iu_k - iu_j - 2)}.\end{aligned} \tag{5.2}$$

or

$$-1 = \frac{Q(iu_k+2)}{Q(iu_k-2)} \Big(\frac{(iu_k-1)}{(iu_k+1)}\Big)^N \tag{5.2'}$$

This is nothing but the Bethe Ansatz equation.
We express this results graphically. Let two boxes correspond to the first and second term in eq.(5.2),

$$\boxed{1}_u = \Big(\frac{(2+u)}{2}\Big)^N \frac{Q(u-1)}{Q(u+1)} \tag{5.3a}$$

$$\boxed{2}_u = \Big(\frac{u}{2}\Big)^N \frac{Q(u+3)}{Q(u+1)}, \tag{5.3b}$$

where the lower index u stresses the u dependendencies of lhs'. We call them "Yangian analogue" to Young tableaux[KS,Suz]. The meaning will become apparent in the following. We draw an arrow to indicate the singularity cancellation,

$$\boxed{1}_u \rightarrow \boxed{2}_u. \tag{5.4}$$

One finds the similarity between the above graph and the structure of A_1 spin 1/2 module; the arrow looks like the action of S^-. In terms of expressions, it means the multiplication by the "lhs" of the Bethe Ansatz equation (5.2') at the singularity points $u = -1 + iu_k$. The $u-$ dependency is easily retrieved by the identification.
We reinterpret the figure as follows. Given the "top" term $\boxed{1}_u$, we like to find the "minimal pole-free set". The rule to generate its descendants is to multiply the expressions by the lhs of Bethe ansatz equation . The resultant expression coincides with $T_1(u) = \boxed{1} + \boxed{2}$

We call this procedure, finding a pole-free set with successive multiplication of the "lhs" of the BAE, as the "Bethe-strap". Up to now, we do not have a proof to justify the "Bethe-strap" method. It, however, gives us correct solutions as far as we can compare with known results. We will see some examples in the following. Moreover, the "Bethe-strap" method gives us the conjectures for the cases with which other methods can not deal.
Let us consider a less trivial example, a model of which the quantum space is again N fold tensor product of two dimensional spaces while the auxiliary space is three dimensional. We know from the item 3 in the section 5,

$$T_2(u) \in T_1(u-1)T_1(u+1). \tag{6.5}$$

Thus we identify the "top" term of the lhs as

$$\boxed{1}_{u-1} \times \boxed{1}_{u+1} \equiv \boxed{1\,|\,1}. \tag{6.6}$$

The Bethe-strap procedure gives the minimal pole-free set,

$$\boxed{1\,|\,1} \to \boxed{1\,|\,2} \to \boxed{2\,|\,2}, \tag{6.7}$$

which agrees with the result in ref[KR]. The figure coincides with that for three dimensional representation of A_1, except for the $u-$ dependency. Indeed, one proceeds further to $m+1$ dimensional case, m arbitrary. The resultant tableaux are exactly those for A_1 $m+1$ dimensional representation with shifts in spectral parameters, $u-m+1, u-m+3, \cdots u+m-1$ from left to right. This coincidence comes from only the condition of singularity cancellation.
The situation holds also good for A_r vertex models, r general. For definiteness we assume the quantum space is given by the N fold tensor of V_{Λ_p}. We prepare some symbols. Let $r+1$ boxes be

$$\begin{aligned} \boxed{a} &= \psi_a(u)\frac{Q_{a-1}(u+a+1)Q_a(u+a-2)}{Q_{a-1}(u+a-1)Q_a(u+a)} \qquad 1 \le a \le r+1 \\ \psi_a(u) &= \begin{cases} ((u+2)/2)^N & \text{for } a \le p \\ (u/2)^N & \text{for } a > p \end{cases} \end{aligned} \tag{6.8}$$

where $Q_0(u) = Q_{r+1}(u) = 1$, while $Q_a(u) \equiv \prod_j (u - iu_j^{(a)}), (a = 1, \cdots, r)$ solves the nested Bethe ansatz equation:

$$-1 = \frac{(iu_k^{(a)} - \delta_{ap})}{(iu_k^{(a)} + \delta_{ap})} \prod_{b=1}^{r} \frac{Q_b(iu_k^{(a)} + (\alpha_a \mid \alpha_b))}{Q_b(iu_k^{(a)} - (\alpha_a \mid \alpha_b))}, \qquad a = 1, \cdots, r. \tag{6.9}$$

The upper index a is referred to as a "color".

As before, we start from the model of which the auxiliary space is the most fundamental one, V_{Λ_1}. The minimal pole-free set is,

$$T'_{\Lambda_1}(u) = \sum_{a=1}^{r+1} \boxed{a}. \tag{6.10}$$

Comparing the result with that from the algebraic Bethe ansatz, we find eq(6.10) is nothing but the eigenvalue of the transfer matrix, i.e., $T'_{\Lambda_1}(u) = T_{\Lambda_1}(u)$. Here we put a prime to distinguish the combinatorial quantity from "true" eigenvalues of transfer matrix.
The singularity cancellation can be depicted as,

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{r} \boxed{r+1}. \tag{6.11}$$

A letter on an arrow signifies which color of singularities two boxes cancel each other. Now we consider "two" box cases. Eq(4.8) suggests that any pair of boxes with spectral parameter $u \pm 1$ may be devided into two groups, $T'_{2\Lambda_1}$ and T'_{Λ_2}. Again, primes stress that they will be defined by box combinatorics. As in $A_1^{(1)}$ case, we expect that any element in $T'_{2\Lambda_1}$ is expressible in a form;

$$\boxed{i}\boxed{j}. \tag{6.12}$$

Similarly, we describe elements in T'_{Λ_2} by two boxes arranged vertically,

$$\begin{array}{|c|}\hline j \\ \hline i \\ \hline\end{array}. \tag{6.13}$$

Here the upper (lower) box carries the spectral parameter $u+1, (u-1)$.
Now we start the Bethe-strap procedure assuming the top term $\boxed{1}\boxed{1}$. The resultant minimal pole free set is given by eq(6.12) with $i \leq j$. This means that any table in eq (6.13) should satisfy $j < i$.
We can also reach the same result starting from the top term,

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}. \tag{6.14}$$

Note two important ingredients,

1 a box with letter i has singularities of colors $i-1$ and i.

2 Thus the table like (6.13) contains singularities in colors $i-1, i, j-1, j$ in general. There are, however, some exceptions: when $j = i-1$, it possesses $j-1, j+1$ color singularities only.

Then it is obvious that the figure for the Bethe-strap procedure coincides with the weight space figure for the highest weight module V_{Λ_2}. The situation is quite the same for V_{Λ_a} for $a \leq r$ for $A_r^{(1)}$. We summarize the Bethe-strap result for the tableaux description for $A_r^{(1)}$.

Rule for the A_r model. *Prepare semi-standard tableaux for $m\Lambda_1$. We assign spectral parameters $u-m+1,\cdots,u+m-1$ from the left to the right. We regard each boxes as expressions under the identification (5.7) and take the product of them. After summing the resultant expressions over all semi-standard tableaux, we have, say, $T'_{m\Lambda_1}(u)$. Similarly for $T'_{\Lambda_a}(u)$ we prepare a set of tableaux made of boxes arranged vertically, with spectral parameters, $u+a-1,\cdots,u-a+1$ from the top to the bottom.*

One can easily prove their pole-freeness. The meanings of functions $x_a^A(u)$ etc.,in eqs (3.1) and (3.2) are now clear; $x_a^A(u) = \boxed{a}_u$.
Our conjecture is,

Conjecture 1. *$T'_{m\Lambda_1}(u)$ and T'_{Λ_a} coincide with the eigenvalues of transfer matrices whose auxiliary spaces are isomorphic to $V_{m\Lambda_1}$ and V_{Λ_a} , respectively.*

We employed the Bethe-strap procedure for some skew Young diagram cases and obtain a generalization of the conjecture 1 as follows. Prepare a skew Young diagram. We assign a number $\in \{1,2,\cdots,r\}$ to a box according to the "semi-standard" rule. That is, the numbers should be strictly increasing from top to bottom and weakly increasing from left to right. We assign the spectral parameters to the boxes which are decreasing by 2 from top to bottom and increasing by 2 from left to right. Through the procedure described above, we have an expression, $T'_{\lambda\subset\mu}$ for a given skew Young diagram $\lambda \subset \mu$.

Conjecture 2. *The expression obtained from the above combinatorial rule constitutes a minimal pole-free set, and it coincides with the eigenvalue of transfer matrix acting on auxiliary space $\lambda \subset \mu$.*

We have different descriptions for the rules imposed for the B_r type case. See [KS, KOS].
For the $A-$ model, there are some supporting arguments available in [Che, Baz-Resh]. The author, however, does not know the explicit proof for the above two conjectures.

6 Equivalence between combinatorial and determinantal expressions

The following theorem is the main body in this section.

Theorem 1. *Combinatorial expressions for transfer matrices coincide with the determinantal expressions given in (3.6),(3.8).*

We have argued in the previous section that the pole-freeness leads to the combinatorial rules for tableaux. Then theorem 1 further states that the pole-free condition fixes a solution to $T-$ system being a determinantal form. And this is our main message in this report.
Though theorem 1 can be proved in a general setting, we prefer to adopt again a simple example.
We first prepare some lemmas.

Lemma 1. *There exists a one to one mapping between a pair of one column Young tableaux* $\{T_{(1^a)}, T_{(1^b)}\}, (a \geq b+2)$ *and a skew Young table* $\lambda \subset \mu$ *s.t.,* $\mu'_1 = a-b-2, \mu'_2 = 0, \lambda'_1 = \lambda_2 = a-1$ *which breaks the horizontal semi-standard rules. Here the semi-standard condition for the vertical adjacent pairs should be valid.*

Proof: We can explicitly construct the map. Let the content of the left(right) column of the skew tableaux be $\{j_1, j_2, \cdots, j_{b+1}\}$ ($\{i_1, i_2, \cdots, i_{a-1}\}$). Looking from the top, we can identify the first adjacent pair which breaks the rule: $j_k > i_{a-b-2+k}$. Then the associated pair of columns is

$$\{(i_1, i_2, \cdots, i_{a-b-2+k}, j_k, j_{k+1}, \cdots, j_{b+1}), (j_1, j_2, \cdots, j_{k-1}, i_{a-b-1+k}, i_{a-b+k}, \cdots, i_{a-1})\}$$

One can similarly construct the inverse map.□
We generalize the result to $(m-1,1)$ columns case. Let $(\lambda \subset \mu)_k$ be a skew Young diagram, $\lambda = ((m-1)^{a+1}), \mu = (m-k-1)$. We fix the spectral parameter of the bottom-right box to be $u-a+m-2$. All numbers assigned to a table satisfy the semi-standard conditions. We introduce a column (1^{a-k}) whose bottom box possesses the spectral parameter $u-a+m$. Let us denote by $TO_{(m^a)}(u,k)$ the set of tableaux of which the first k adjacent columns from the right break the horizontal condition. Here the bottom-right box is assigned the spectral parameter $u-a+m$.
We have

Lemma 2. *There is a one-to-one correspondence between a pair of tableaux* $\{(\lambda \subset \mu)_k, (1^{a-k})\}$ *and a member in* $TO_{(m^a)}(u,k-1) \cup TO_{(m^a)}(u,k)$.

The proof can be completed with repeated applications of the lemma 1 with some "end" conditions.
Summing up them with alternating signs, we have

Theorem 2.

$$\sum_{k=0}^{a}(-1)^k T'_{(\lambda\subset\mu)_k} T'_{(1)^{a-k}} = T'_{(m^a)}.$$

To prove the theorem 1, it is useful to adopt the induction method w.r.t. width of the Young diagram m. We have to prepare the analogue of theorem 2 where the rhs is the T' for a general Young diagram. The proof for such general case can be done in a straightforward way, but lengthy. Thus we will not present it here and assume that our theorem holds good for any skew Young diagram of width $m-1$.
Let us expand the determinantal expression for $T_{(m^a)}(u)$ w.r.t. the $m-$ th column and compare the result with theorem 2. Especially, consider the $(m-k,m)$ minor. The $(m-k,m)$ element is $T^{a-k}(u+m-k-1)$ and is nothing but the $T'_{(1^{a-k})}$. Close examination reveals the minor coincides with $T'_{(\lambda\subset\mu)_k}$ under the induction assumption. Therefore, the determinantal expression and the combinatorial one coincide at this stage. One can argue a general skew case in a similar way, which completes the proof of theorem 1.$\square$.

7 Conclusion

In this report, we have presented a solution to a certain discretized Toda field equation. The solution is inspired by the studies on two dimensional solvable lattice models. In the view of the latter, the determinantal structure of the solution can be easily understood as a consequence of pole-freeness. We might expect DPEs also admit such interpretation in their structure of solution and the singularity confinement property.

references

[Bax] R.J. Baxter, Ann. Phys. **70** (1972) 193

[Baz-Res] V.V.Bazhanov and N.Yu Reshetikhin, J. Phys. A. **23** (1990) 1477.

[Che] I. Cherednik, in Proc XVII Int. Conf. on Differential Geometric Methods in Theor. Phys. ed A.I. Solomon (World Scientific 1989)

[EH] E. Lieb, Phys. Rev. **162** (1967) 162.

[GRP] B.Grammaticos, A. Ramani and V. Papageorgiou, Phys. Rev. Lett. **67** (1991) 1825.

[Kaji] K. Kajiwara, talk at the meetimg of the Physical Society Japan, at Kanagawa University (1995) and the proceeding in this volume.

[KNH] A. Kuniba, S. Nakamura and R. Hirota, hep-th 9509039.

[KNS1] A. Kuniba, T. Nakanishi and J. Suzuki, Int. J. Mod. Phys. A **9**(1994) 5215.

[KOS] A. Kuniba, Y. Ohta and J. Suzuki, hep-th. 9506167, J. Phys. A., in press.

[KR] A.N. Kirillov and N. Yu Reshetikhin, J.Phys.A. **20**(1987) 1565, ibid 1587.

[KS] A. Kuniba and J. Suzuki, hep-th.9406180 , C.M.P. in press.

[Mac] I.M. Macdonald, Symmetric functions and Hall polynomials. (Clarendon press, Oxford, 1979)

[Okado] M. Okado, C.M.P. **134** (1990) 467.

[Res] N.Yu Reshetikhin, JETP **57** (1983) 691.

[RGH] A.Ramani, B.Grammaticos and J.Hietarinta, Phys. Rev. Lett . **67** (1991) 1829.

[Suz] J. Suzuki, Phys. Lett. A **195** (1994) 190.